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Note

An Approximate Method for Evaluating the Ratio of Two Complete Elliptic Integrals of the First Kind

In calculations using Schwartz–Christoffel conformal transformations the ratio $K'(k)/K(k)$ frequently arises. $K(k)$ is the complete elliptic integral of the first kind of order k and $K'(k) = K((1-k^2)^{1/2})$. This ratio or an exponential version of it called the nome, is tabulated in standard references such as Abramowitz and Stegun 1968 [1]. Alternatively, numerical methods exist by which it may be calculated. It is useful, however, to have a simple formula available which is accurate enough for most purposes and can be used to generate any required value over the whole range $0 < k < 1$.

Such a formula may be derived in the following way. For small values of k [1]

$$K'(k)/K(k) \approx \frac{4}{\pi} \ln(2/k^{1/2}). \tag{1}$$

A transformation exists [2], however,

$$K'(k)/K(k) = 4K(k_1)/K'(k_1), \tag{2}$$

where

$$k_1^{1/2} = (1-k^{1/2})/(1+k^{1/2})$$

which combined with Eq. (1) gives

$$K'(k)/K(k) \approx \pi/\ln(2(1+k^{1/2})/(1-k^{1/2})) \tag{3}$$

This formula is accurate to three parts of a million in the range $0.7071 \leqslant k < 1$. A better formula than Eq. (1) which is valid to the same degree of accuracy over the range $0 < k \leqslant 0.7071$ is obtained by using the relationship

$$K'(k) = K((1-k^2)^{1/2}).$$

Then, defining $k' = (1-k^2)^{1/2}$, for $0 < k \leqslant 0.7071$,

$$K'(k)/K(k) \approx (1/\pi)\ln(2(1+k'^{1/2})/(1-k'^{1/2})). \tag{4}$$

Inverse formulae to Eqs. (3) and (4) can easily be obtained, again, accurate to three parts in a million.

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For $0 < K'(k)/K(k) \leqslant 1$

$$k \approx \{(\exp(\pi K(k)/K'(k)) - 2)/(\exp(\pi K(k)/K'(k)) + 2)\}^2 \tag{5}$$

and for $K'(k)/K(k) \geqslant 1$

$$k \approx \frac{4 \exp(\pi K'(k)/2K(k))[\exp(2\pi K'(k)/K(k)) + 4]^{1/2}}{[\exp(\pi K'(k)/K(k)) + 2]^2}. \tag{6}$$

The accuracy of these approximations can be seen by considering the worst case, when $k = 0.7071068$. For this case Eq. (3) or (4) yield

$$K'(k)/K(k) = 0.9999978 \qquad \text{or} \qquad 1.0000022 \qquad (\text{tabulated value} = 1).$$

Alternatively, if $K'(k)/K(k) = 1$, Eq. (5) or (6) produce

$$k = 0.7071051 \qquad \text{or} \qquad 0.7071085 \qquad (\text{tabulated value} = 0.7071068).$$

The only other limitation to the use of these formulae is the loss of significant figures caused by the precision of the computer used. This will cause trouble in Eq. (3) when k is very close to 1 and in Eq. (4) when k is very close to 0. The problem, however, can be largely avoided by rewriting Eqs. (3) and (4) as follows: For

$$0.7071 < k \leqslant 1, \qquad K'(k)/K(k) \approx \pi/\ln(2(1 + k^{1/2})^2/(1 - k)) \tag{7}$$

whilst for

$$0 < k \leqslant 0.7071, \qquad K'(k)/K(k) \approx (1/\pi) \ln(2(1 + k'^{1/2})^2(1 + k')/k^2). \tag{8}$$

References

1. M. Abramowitz and I. A. Stegun (Eds.), "Handbook of Mathematical Functions," Dover, New York, 1968.
2. P. Henrici, "Applied and Computational Complex Analysis," Vol. I, Wiley–Interscience, New York, 1974.

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